CHAPTER V

THE SIMPLE WAVE EQUATION

5·1. Invariant Equations.

In order that the laws of physics may be independent of the choice of frame (among the equivalent orthogonal frames) they must be expressible as tensor equations. In wave-tensor calculus, the simplest non-trivial tensor equation is of the form

$$H\psi = 0, \tag{5·11}$$

where H is a mixed wave tensor and ψ a covariant wave vector. Then (5·11) is a vector equation $H_\alpha{}^\beta \psi_\beta = 0$ equivalent to four algebraic equations.

It may be anticipated that the simplest, and presumably the most fundamental, laws of physics will have this form. Alternatively, regarding (5·11) as a definition rather than a law, it is an appropriate means of introducing a wave vector ψ and relating it to the ordinary space vectors of physics. For we have seen (§ 4·4) that a mixed wave tensor H is constituted of space tensors which will presumably be recognised as such in our practical observations; but there is no such "projection" of ψ into a space-time representation, and its connection with the ordinary space tensors of physics can only be expressed indirectly by an equation such as (5·11).

It is appropriate to introduce simultaneously a contravariant wave vector χ^*, satisfying

$$\chi^* H = 0. \tag{5·12}$$

We may expect that ψ and χ^* will occur symmetrically in physical theory.

Thus far our argument has been that if ever nature condescends to simplicity, equations of the types (5·11) and (5·12) will figure in her scheme.

Before the birth of wave mechanics the systematised part of physics was wholly described by space vectors and tensors. Wave mechanics introduced a new kind of entity ψ. It was introduced in the way here proposed by a "wave equation" in which the coefficients were the ordinary space-tensor quantities of physics. The original ψ of Schrödinger did not satisfy the relativity requirements of atomic physics; but in 1928 Dirac introduced a ψ with four components, which satisfied an equation invariant for the six relativity rotations of space-time, although the invariance was not of a kind contemplated in the usual tensor calculus. Our anticipated fundamental equation turns out to be a form of Dirac's equation.†

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† The form given by Dirac (*Quantum Mechanics*, 2nd ed., p. 255, equations (9) and (10)) is the equivalent strain vector equation, which we shall obtain in (7·73). Dirac further postulates, as a "reality condition", that the two wave vectors are conjugate complex quantities. Our reality conditions are determined directly from relativity principles in Chapter VI, and do not impose this restriction.

The mixed wave tensor H used in Dirac's equation is limited to five components. One component has dropped out through special choice of axes, permissible when as usual the region contemplated is small enough to be treated as flat. Apart from this the truncation is significant, because the general mixed wave tensor T cannot be reduced to Dirac's special form H by any choice of axes. We shall account for this limitation in § 5·4.

We shall derive Dirac's equation according to the principles which we are developing in § 5·4; but we shall first examine its elementary properties—looking ahead to see the theory which we are about to meet.

Dirac's equations are $$H\psi=0, \quad \chi^*H=0, \tag{5·13}$$
where $$H=E_1p_1+E_2p_2+E_3p_3+E_4p_4-m \tag{5·14}$$
and E_1, E_2, E_3, E_4 constitute a tetrad. We call H the *hamiltonian*.[†] Since H is required to be a mixed wave tensor, its components form space tensors in accordance with the specification in (4·42), namely (p_1, p_2, p_3, p_4) is a space vector and the quarterspur $-m$ is an invariant.

Conversely if, following Dirac, we construct a hamiltonian H out of a "momentum vector" (p_1, p_2, p_3, p_4) and an invariant mass m by the formula (5·14), H will be a mixed wave tensor; and therefore the wave equations $H\psi=0$, $\chi^*H=0$ will be tensor equations which continue to be satisfied when any of the six relativity transformations of space-time are applied.

Thus the invariance of Dirac's equation for relativity transformations, which was a novel kind of invariance from the point of view of ordinary tensor calculus, is an elementary consequence of wave-tensor calculus.

5·2. Properties of Dirac's Equation.

The equation $H\psi=0$ shows that H has an eigenvalue 0, and hence that it is singular (§ 3·7 (*b*)). It has a pseudo-reciprocal
$$H'=E_1p_1+E_2p_2+E_3p_3+E_4p_4+m. \tag{5·21}$$
For, multiplying by (5·14),
$$HH'=-p_1^2-p_2^2-p_3^2-p_4^2-m^2. \tag{5·22}$$
The product contains no non-algebraic terms, and therefore vanishes by (2·66).

For a physically real momentum vector (p_1, p_2, p_3, p_4), p_1, p_2, p_3 are real and p_4 is imaginary. Let
$$p_4=ip_0, \tag{5·23}$$

[†] In classical theory the hamiltonian is the expression for the energy $(-ip_4)$ in terms of the momenta p_1, p_2, p_3 and coordinates. If, following the usual relativistic view, time is treated on the same footing with the other coordinates, the hamiltonian is correspondingly defined as the expression for the proper energy m in terms of p_1, p_2, p_3, p_4 and the coordinates. This would make the hamiltonian strictly $H+m$, but we shall use the term without regard to an additive constant.

so that p_0 is the real time component, i.e. the energy or mass. The vanishing of (5·22) gives

$$m^2 = p_0{}^2 - p_1{}^2 - p_2{}^2 - p_3{}^2. \tag{5·24}$$

This identifies ± m with the proper energy (or proper mass) corresponding to the momentum vector. Thus m must be real.

By (5·24)

$$p_0 = (m^2 + p_1{}^2 + p_2{}^2 + p_3{}^2)^{\frac{1}{2}}. \tag{5·251}$$

Usually p_1, p_2, p_3 are small compared with m, and we then have the "classical" approximation for the energy

$$p_0 = m + (p_1{}^2 + p_2{}^2 + p_3{}^2)/2m. \tag{5·252}$$

The general solution of the wave equations is found as follows. Let ϕ, ω^* be arbitrary four-valued quantities. Since $HH' = H'H = 0$, we have

$$H(H'\phi) = 0, \quad (\omega^* H')H = 0.$$

Thus

$$\psi = H'\phi, \quad \chi^* = \omega^* H' \tag{5·26}$$

are solutions of (5·13). Inserting row-and-column suffixes, these become

$$\left.\begin{aligned} \psi_\alpha &= H_{\alpha 1}{}'\phi_1 + H_{\alpha 2}{}'\phi_2 + H_{\alpha 3}{}'\phi_3 + H_{\alpha 4}{}'\phi_4, \\ \chi_\alpha &= \omega_1 H_{1\alpha}{}' + \omega_2 H_{2\alpha}{}' + \omega_3 H_{3\alpha}{}' + \omega_4 H_{4\alpha}{}'. \end{aligned}\right\} \tag{5·27}$$

Since ϕ_1, ϕ_2, ϕ_3, ϕ_4 are arbitrary coefficients, the general value of ψ_α is a linear combination of four elementary solutions $H_{\alpha 1}{}'$, $H_{\alpha 2}{}'$, $H_{\alpha 3}{}'$, $H_{\alpha 4}{}'$, i.e. the four rows of the matrix H'. Similarly the four columns of H' are the elementary solutions for χ_α. Since H' is singular, its determinant vanishes, and therefore only three of the rows and three of the columns are linearly independent. There are therefore not more than three independent solutions in each case.

For example, use the matrix representation (3·61). The special form (5·21) for H' gives

$$-iH' = \begin{matrix} p_2 - im & p_1 + p_0 & 0 & p_3 \\ p_1 - p_0 & -p_2 - im & -p_3 & 0 \\ 0 & -p_3 & p_2 - im & p_1 - p_0 \\ p_3 & 0 & p_1 + p_0 & -p_2 - im. \end{matrix} \tag{5·28}$$

Any row gives a solution for ψ and any column a solution for χ^*. It is not necessary to choose the solutions in a corresponding way.† For example, we might take

$$\psi = (p_2 - im, p_1 + p_0, 0, p_3), \quad \chi^* = (0, -p_3, p_2 - im, p_1 + p_0)$$

as the pair of wave vectors constituting a solution of the wave equation.

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† Our treatment here differs fundamentally from that of Dirac. See footnote, p. 62; also § 8·6.

5·3. The Stream Vector.

Let ψ, χ^* be solutions of the equations $H\psi=0$, $\chi^*H=0$; and let

$$\psi\chi^*=J=\Sigma j_\mu E_\mu. \tag{5·31}$$

Multiply the wave equations by initial χ^*E_{12} and final $E_{12}\psi$, respectively; we obtain

$$\chi^*(E_2p_1-E_1p_2+E_{12}E_3p_3+E_{12}E_4p_4-E_{12}m)\psi=0,$$

$$\chi^*(-E_2p_1+E_1p_2+E_3E_{12}p_3+E_4E_{12}p_4-E_{12}m)\psi=0.$$

Hence, subtracting,

$$2\chi^*(E_2p_1-E_1p_2)\psi=0,$$

so that, by (3·37),

$$j_2p_1-p_1j_2=0. \tag{5·321}$$

Again, multiplying by initial χ^*E_{15} and final $E_{15}\psi$, and subtracting,

$$\begin{aligned}0&=\chi^*(E_{15}H-HE_{15})\psi\\&=2\chi^*(E_5p_1)\psi=-8j_5p_1,\end{aligned} \tag{5·322}$$

so that $j_5=0$. Also multiplying by initial χ^*E_1 and final $E_1\psi$, and adding,

$$\begin{aligned}0&=\chi^*(E_1H+HE_1)\psi\\&=2\chi^*(-p_1-E_1m)\psi\\&=-8(ip_1j_{16}-j_1m).\end{aligned} \tag{5·323}$$

The results (5·321), (5·322), (5·323) give

$$\frac{j_1}{p_1}=\frac{j_2}{p_2}=\frac{j_3}{p_3}=\frac{j_4}{p_4}=\frac{j_5}{0}=\frac{ij_{16}}{m}. \tag{5·33}$$

The wave equation can therefore be written in the equivalent form

$$(E_1j_1+E_2j_2+E_3j_3+E_4j_4-E_{16}j_{16})\psi=0. \tag{5·34}$$

We call (j_1, j_2, j_3, j_4) the *stream vector*. The whole set of sixteen j_μ is the *complete stream vector*. We have here proved that the stream vector is equal to the momentum vector except for a numerical factor. Multiplying the complete stream vector by the same factor we obtain the *complete momentum vector*.

We regard this correspondence of the stream vector and momentum vector as a coalescence which occurs in the peculiarly simple system here studied. In more general physical systems they are not so closely connected. According to the definition of the momentum vector usually adopted in quantum theory and reached later in this book, the components p_μ are not necessarily algebraic quantities; they may be matrices or general symbols. On the other hand the components j_μ of the stream vector are necessarily algebraic quantities.

The result (5·34) leads us to a new view of the wave equation. Consider a physical system described by a *pure* (i.e. factorisable) wave tensor J, or by the equivalent set of space tensors. We are not given the whole set of space tensors, but only one of the space vectors (j_1, j_2, j_3, j_4) together with the two invariants j_5 $(=0)$ and j_{16}. We cannot therefore determine definitely the factors of J; but our data are sufficient to limit them to certain possibilities, viz. they must be solutions of (5·34) and of the corresponding equation for χ^*.

The wave equation is therefore an equation for determining the possible factors of a wave tensor, which is only partly known.

At this point we must try to make clear a difference in our attitude towards wave mechanics from that which appears to be usual among quantum physicists. It will probably be agreed that wave mechanics is a *method* of analysis, not a *theory* of phenomena. The ψ waves have no objective existence; we invent them as required in solving our problems. In the present treatment we have found that any space vector can be expressed as a wave tensor. Ordinarily it is not a pure wave tensor; but it can be represented as a sum of pure wave tensors, which are then resolved into their wave-vector factors. If the space vector is a function of the coördinates, the wave-vector factors become "wave functions". In this way wave functions appear in connection with any characteristic of a system which is described by space vectors. It is therefore ambiguous to speak of the wave functions of a system; we should rather speak of the wave functions associated with some specified tensor of the system. Reference to the wave function or the wave equation of a system leaves us in the same state of conjecture as if reference were made to "the tensor of the hydrogen atom" or "the equation of the sun".

I am not cavilling at expressions, whose meaning is doubtless made plain either by the context or by custom. My point is that when wave analysis is our standard procedure—when the ordinary tensor calculus is replaced by wave-tensor calculus—we shall introduce new wave functions as casually as we introduce new tensors. The domain of physics treated in this book is for the most part different from that which has occupied the attention of writers on pure quantum theory. Sometimes our wave functions will coincide with theirs; sometimes they will differ. We find it well to maintain a certain amount of contact in order to utilise well-known results; but in principle we do not bind ourselves to use the wave functions that the quantum physicists have discussed. Remembering that the introduction of wave functions is merely a factorisation, we must obviously retain freedom to employ factorisation whenever it is useful.

The reader must therefore be prepared to find here a greater elasticity in the definition and use of wave functions than he has been accustomed to.

5·4. The Wave Equation as an Identity.

If we represent E_1, E_2, E_3, E_4, E_5 by the special pentad of matrices (3·27), it is not difficult to prove by straightforward verification that

$$\sum_{\mu=1}^{\mu=5} (E_\mu \psi)_\alpha (E_\mu \psi)_\beta - (E_{16}\psi)_\alpha (E_{16}\psi)_\beta \equiv 0, \qquad (5·41)$$

where ψ is any four-valued quantity. Here, as usual, $E_\mu \psi$ is the four-valued quantity formed by chain multiplication, and $(E_\mu \psi)_\alpha$ is one of its four components, i.e. $(E_\mu \psi)_\alpha = (E_\mu)_{\alpha\beta} \psi_\beta$.

Any other pentad E_μ' is obtained by a transformation $E_\mu' = qE_\mu q^{-1}$. Let $\psi' = q\psi$; then $E_\mu' \psi' = qE_\mu \psi$, so that

$$\begin{aligned}(E_\mu' \psi')_\sigma (E_\mu' \psi')_\tau &= (qE_\mu \psi)_\sigma (qE_\mu \psi)_\tau \\ &= q_{\sigma\alpha} (E_\mu \psi)_\alpha q_{\tau\beta} (E_\mu \psi)_\beta.\end{aligned}$$

Hence, multiplying (5·41) by $q_{\sigma\alpha} q_{\tau\beta}$, we have

$$\sum_{\mu=1}^{\mu=5} (E_\mu' \psi')_\sigma (E_\mu' \psi')_\tau - (E_{16} \psi')_\sigma (E_{16} \psi')_\tau = 0. \tag{5·42}$$

We can choose ψ' arbitrarily; because (since q is not singular) the corresponding $\psi = q^{-1} \psi'$ can be employed in (5·41).

Thus the identity (5·41), verified for a particular pentad, is true for any pentad of matrices whatsoever.

Let χ^* be another arbitrary four-valued quantity. Multiply (5·41) by initial χ_α (inner multiplication). We have by (3·37)

$$\chi_\alpha (E_\mu \psi)_\alpha = \chi^* E_\mu \psi = -4 j_\mu,$$

where $\psi \chi^* = J = \Sigma j_\mu E_\mu$. The result is therefore

$$\sum_1^5 j_\mu (E_\mu \psi)_\beta - j_{16} (E_{16} \psi)_\beta = 0$$

or

$$(E_1 j_1 + E_2 j_2 + E_3 j_3 + E_4 j_4 + E_5 j_5 - E_{16} j_{16}) \psi = 0. \tag{5·43}$$

We can show similarly that

$$\chi^* (E_1 j_1 + E_2 j_2 + E_3 j_3 + E_4 j_4 + E_5 j_5 - E_{16} j_{16}) = 0. \tag{5·44}$$

Except that the term in j_5 is included, these are the wave equations as given in (5·34). They are here obtained as an identity satisfied by any two wave vectors ψ, χ^* and their outer product J.

We see that Dirac was right in restricting his hamiltonian to the above terms, instead of employing the sixteen terms of a complete space vector. By omitting j_5 he restricts the equation to stream vectors which have zero component normal to space-time; otherwise his equation is a perfectly general one satisfied by the factors of any pure wave tensor. The hamiltonian H is part of the complete stream vector J, except that the sign of j_{16} is reversed. It must not be supposed that the components of J (other than j_5) which do not appear in H are zero.

On the other hand Dirac's postulate that ψ and χ^* (or rather a quantity ϕ^* easily derived from χ^*) are conjugate complex quantities would restrict

J to very special forms. The restriction has to do with certain special applications, and is inappropriate in general theory.

Multiply (5·43) by initial χ^* and apply (3·37). We obtain

$$j_1^2+j_2^2+j_3^2+j_4^2+j_5^2-j_{16}^2=0. \qquad (5{\cdot}45)$$

Again, multiply (5·43) by initial $\chi^* E_5$ and apply (3·37). We obtain

$$j_1 j_{15}+j_2 j_{25}+j_3 j_{35}+j_4 j_{45}=0. \qquad (5{\cdot}46)$$

These, and the corresponding equations obtained by substituting other pentads, are relations satisfied identically by the components of a pure wave tensor. There are, of course, no such relations between the components of a general wave tensor T which is not stated to be factorisable.

Let $$P_\alpha=\sum_{\mu=0}^{\mu=5} j_{\alpha\mu}E_{\alpha\mu} \quad (\alpha=0,\ 1,\ 2,\ 3,\ 4,\ 5;\ \mu\neq\alpha). \qquad (5{\cdot}47)$$

For fixed α, the matrices $E_{\alpha\mu}$ form a pentad. We therefore call P_α a *pentadic part* of J. There are six pentadic parts which overlap, so that

$$\tfrac{1}{2}\Sigma_\alpha P_\alpha+\mathrm{qs}\,J=J. \qquad (5{\cdot}48)$$

The pentad which we have been using corresponds to $\alpha=0$, and the wave equation (5·43) can be written

$$(P_0-ij_{16})\,\psi=0.$$

But since the proof holds for any pentad, we have more generally

$$(P_\alpha-ij_{16})\,\psi=0 \qquad (5{\cdot}491)$$

or equivalently $$P_\alpha\psi=(\mathrm{qs}\,J)\,\psi \qquad (5{\cdot}492)$$

for all six values of α.

From the present standpoint the use of the wave equation is to determine the factors of a pure wave tensor J. It seems to be generally true that in physics we determine a factor ψ, not because its value is of particular importance to us, but because that happens to be the most convenient way of ascertaining that a factor exists, i.e. that J is pure. For example, the wave function ψ of a hydrogen atom is investigated primarily because the mere existence of such a function imposes certain conditions on the hamiltonian (which is part of J), and these conditions determine the energy levels of the atom. It would probably be difficult to solve the more complex problems of quantum theory without evaluating ψ; but since the observables of physics are always space tensors and therefore derived from wave tensors of the second (or higher) rank, the wave vector factors must ultimately be recombined.

In the present case (which is perhaps too elementary to be typical) the conditions for purity of J are expressed directly by the equations (5·45) and (5·46), and there is no need to evaluate the factors.

5·5. Standard Forms of Pure Wave Tensors.

Equation (5·492) asserts:

A factor of J is an eigensymbol of every pentadic part of J, and the eigenvalue of a pentadic part is qs J. (5·51)

Consider an antitriad $E_{\mu\nu}$, $E_{\sigma\tau}$, $E_{\lambda\rho}$ (2·36). Each pentad contains one and only one member of an antitriad. Hence, in the expression $(E_{\mu\nu}+E_{\sigma\tau}+E_{\lambda\rho})\,m$, each term is a pentadic part and has eigenvalues $\pm im$. Accordingly the form

$$J=(\pm E_{\mu\nu}\pm E_{\sigma\tau}\pm E_{\lambda\rho}+E_{16})\,m \tag{5·52}$$

will, if the signs are properly chosen, satisfy the condition (5·51) that the eigenvalue of every pentadic part is equal to the quarterspur. It turns out that four of the combinations of sign make J factorisable, and four do not. For example, take the + sign for the first two terms; then by (5·492)

$$E_{\mu\nu}\psi=i\psi, \quad E_{\sigma\tau}\psi=i\psi. \tag{5·53}$$

Hence, if μ, ν, σ, τ, λ, ρ is an even permutation of 0, 1, 2, 3, 4, 5,

$$iE_{\lambda\rho}\psi=E_{\mu\nu}E_{\sigma\tau}\psi=i^2\psi,$$

so that the + sign must also be taken for the third term in order to satisfy (5·492). It is then easy to verify that J is factorisable by working out the factors in a particular matrix representation, or by testing it for idempotency according to the theory given in the next section.

Accordingly our result is that

$$J=(E_{\mu\nu}+E_{\sigma\tau}+E_{\lambda\rho}+E_{16})\,m \tag{5·54}$$

is a pure matrix if μ, ν, σ, τ, λ, ρ is an even permutation. Any two of the first three terms can be given negative sign, since this is equivalent to reversing the order of their suffixes and the permutation remains even.

Any non-degenerate pure wave tensor can be reduced to the standard form (5·54) by a relativity transformation $J'=qJq'$. We first make a transformation so that one of the components, say j_5, becomes zero. Then, by (5·45),

$$j_1^{\,2}+j_2^{\,2}+j_3^{\,2}+j_4^{\,2}=j_{16}^{\,2}, \quad j_{15}^{\,2}+j_{25}^{\,2}+j_{35}^{\,2}+j_{45}^{\,2}=j_{16}^{\,2}.$$

Since J is non-degenerate, $j_{16}\neq 0$. Hence the vectors (j_1, j_2, j_3, j_4) and $(j_{15}, j_{25}, j_{35}, j_{45})$ have the same non-zero length, and by (5·46) they are at right angles. We can therefore choose two of the axes in four dimensions to coincide with them; we then have

$$j_1=j_{25}=j_{16}, \quad j_2=j_3=j_4=j_{15}=j_{35}=j_{45}=j_5=0. \tag{5·551}$$

Applying (5·492) with $\alpha=0$, 5, 3, we now have

$$E_1j_1\psi=ij_{16}\psi, \quad E_{25}j_{25}\psi=ij_{16}\psi, \quad (E_{31}j_{31}+E_{32}j_{32}+E_{34}j_{34})\,\psi=ij_{16}\psi. \tag{5·552}$$

Thus ψ is an eigensymbol of the two commuting symbols E_1, E_{25} and therefore of their product iE_{34}. It is therefore an eigensymbol of $(E_{31}j_{31}+E_{32}j_{32})$;

and the eigenvalue must be zero because $(E_{31}j_{31}+E_{32}j_{32})$ anticommutes with E_{34}. Hence the third equation of (5·552) breaks up into

$$(E_{31}j_{31}+E_{32}j_{32})\psi=0, \quad E_{34}j_{34}\psi=ij_{16}\psi,$$

so that

$$j_{31}=\pm ij_{32}, \quad j_{34}=\pm j_{16}.$$

Hence $j_{31}(E_{31}\pm iE_{32})\psi=0$; or, multiplying by E_{31}, $j_{31}(-1\pm iE_{12})\psi=0$. Now ψ cannot be an eigensymbol of E_{12}, because it is an eigensymbol of E_1 which anticommutes with E_{12}. Hence

$$j_{31}=0, \quad j_{32}=0.$$

Similarly we find $j_{41}=0$, $j_{42}=0$.

The pentad $\alpha=1$ then gives $(E_1j_1+E_{12}j_{12})\psi=ij_{16}\psi$. Hence, by the first equation of (5·552), $j_{12}=0$. All the components are now accounted for, and J reduces to $(E_1+E_{25}+E_{34}+E_{16})j_{16}$, which is of the required standard form.

To obtain a standard form for a *degenerate* pure wave tensor we proceed as follows. Let $J=\psi\chi^*$ be a degenerate pure wave tensor ($j_{16}=0$), and let j_ϖ be a component which does not vanish. Then $E_\varpi J$ is a pure wave tensor, since it has factors $E_\varpi\psi$ and χ^*; and it is non-degenerate since its quarterspur is $E_\varpi(E_\varpi j_\varpi)=-j_\varpi$. Hence $E_\varpi J$ can be reduced to the form (5·54) by a relativity transformation. We take therefore

$$E_\varpi J=(E_{\mu\nu}+E_{\sigma\tau}+E_{\lambda\rho}+E_{16})m.$$

E_ϖ cannot be $E_{\mu\nu}$, $E_{\sigma\tau}$, $E_{\lambda\rho}$ or E_{16}, since J would then be non-degenerate; but it can be any other E-symbol. Taking $E_\varpi=E_{\mu\sigma}$, we obtain

$$J=(E_{\nu\sigma}+E_{\mu\tau}+iE_{\nu\tau}+iE_{\sigma\mu})m. \tag{5·56}$$

This is the standard form for a degenerate pure wave tensor.

5·6. Idempotency.

A symbol J is said to be *idempotent* if $J^2=J$.

To *normalise* an E-number we multiply it by an algebraic factor so as to make the quarterspur $\frac{1}{4}$. If it is represented as a matrix (so that a spur exists) we normalise it by making the spur 1. It is, of course, impossible to normalise a degenerate E-number.

We shall show that *a necessary and sufficient condition for a non-degenerate matrix to be pure is that it shall be idempotent when normalised.* (5·61)

Let $J=\psi\chi^*$ be a normalised matrix so that

$$\operatorname{spur} J=\chi^*\psi=1.$$

Then

$$J^2=\psi\chi^*\psi\chi^*=\psi\,.\,1\,.\,\chi^*=J,$$

so that the condition is necessary. To prove that it is sufficient, let T be a matrix satisfying

$$T^2=T, \quad \operatorname{spur} T=1. \tag{5·62}$$

Any matrix can be expressed as the sum of a number of vector products; therefore let

$$T=\psi_a\chi_a^*+\psi_b\chi_b^*+\psi_c\chi_c^*+\dots. \qquad (5\cdot 63)$$

Here the suffixes $a, b, c, \dots$ distinguish different vectors, the row-and-column suffixes being omitted as usual. We write A_{ab} for the scalar product $\chi_a^*\psi_b$, so that the product $\psi_a\chi_a^*\psi_b\chi_b^*$ reduces to $A_{ab}\psi_a\chi_b^*$. Then

$$T^2-T=(A_{aa}-1)\,\psi_a\chi_a^*+A_{ab}\psi_a\chi_b^*+A_{ba}\psi_b\chi_a^*+\dots=0. \quad (5\cdot 64)$$

Corresponding to the four suffixes of χ this gives four linear equations satisfied by the vectors $\psi_a, \psi_b, \psi_c, \dots$. Using any one of these equations to give the value of ψ_a in terms of the other ψ's, we can eliminate ψ_a in (5·63) and so reduce by one the number of vector products on the right-hand side of (5·63). Repeating the process, we reduce the number of vector products one by one.

The procedure fails if the coefficients of ψ_a vanish in all four equations, i.e. if

$$(A_{aa}-1)\,\chi_a^*+A_{ab}\chi_b^*+A_{ac}\chi_c^*+\dots=0. \qquad (5\cdot 65)$$

But we can then use (5·65) to eliminate χ_a^* in (5·63), and the number of vector products is again reduced by one.

The reduction can be continued so long as there are any non-vanishing coefficients in (5·64). When all the coefficients vanish so that

$$A_{aa}=A_{bb}=A_{cc}=\dots=1, \quad A_{ab}=A_{ba}=\dots=0,$$

no further reduction is possible. We then have, by (5·63),

$$\begin{aligned}\text{spur}\,T&=\text{spur}\,\psi_a\chi_a^*+\text{spur}\,\psi_b\chi_b^*+\text{spur}\,\psi_c\chi_c^*+\dots\\&=A_{aa}+A_{bb}+A_{cc}+\dots\\&=1+1+1+\dots.\end{aligned}$$

But spur $T=1$, so that there can be only one term on the right-hand side. That is to say, T is the product of two vectors.

A pure matrix is necessarily singular. This follows from § 3·7 (*b*), since the idempotent condition, $J^2-J=0$, gives eigenvalues 0 and 1. A singular matrix is not necessarily pure.

If the square of a matrix is -1, the question sometimes arises whether it has to fulfil any other condition in order that it may be a member of a complete orthogonal set. The most commonly occurring combinations whose squares are algebraic are triadic and pentadic expressions; these can serve as individual members of a new complete set. But it has been pointed out by D. E. Littlewood† that we can also form combinations of antiperpendicular matrices whose squares are algebraic, and these cannot be members of a complete set. We find by direct multiplication that the square of

$$\tfrac{1}{2}(E_{\mu\nu}+E_{\sigma\tau}-E_{\lambda\rho}+E_{16}) \qquad (5\cdot 66)$$

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† *Journ. Lond. Math. Soc.* **9**, 41 (1934).

is -1. Of the eight possible combinations of sign in an antitetrad, four yield factorisable matrices, as we have seen; the other four give matrices whose squares are algebraic.

The quarterspur of (5·66) is $\frac{1}{2}i$, whereas the quarterspur of E_μ is 0 or i. Since the quarterspur is invariant for the transformation $F_\mu = qE_\mu q'$, there can be no such transformation connecting (5·66) and E_μ. Therefore (5·66) cannot be a member of a complete set. We shall call an expression of the form (5·66), or reducible to it by a relativity transformation, a *compact* E-number.

I have not as yet found any physical application for compact E-numbers; but perhaps others will be more successful. They surely must have an importance of some kind, possibly in the theory of radiation or even in the theory of the nucleus—subjects which we do not seriously attempt to treat in this book.

5·7. Spectral Sets.

We suggested in § 5·4 that the wave vector ψ was investigated in physics, not for its own sake, but because the existence of factors imposes certain invariant conditions on the stream vector and on the hamiltonian which forms part of it. We may now go a step further, and say that the condition which it is sought to impose is that of idempotency. Those familiar with the Group Theory of wave mechanics will recall the fundamental part played by idempotent operators in selecting the "pure" states of a statistical *ensemble*.

Consider the wave tensors

$$\left.\begin{aligned} J_a &= -\tfrac{1}{4}i\,\,(E_{\mu\nu}+E_{\sigma\tau}+E_{\lambda\rho}+E_{16}),\\ J_b &= -\tfrac{1}{4}i\,(-E_{\mu\nu}-E_{\sigma\tau}+E_{\lambda\rho}+E_{16}),\\ J_c &= -\tfrac{1}{4}i\,(-E_{\mu\nu}+E_{\sigma\tau}-E_{\lambda\rho}+E_{16}),\\ J_d &= -\tfrac{1}{4}i\,\,(E_{\mu\nu}-E_{\sigma\tau}-E_{\lambda\rho}+E_{16}).\end{aligned}\right\} \tag{5·71}$$

We have found in (5·54) that these are pure. Since the quarterspur is $-\frac{1}{4}iE_{16}=\frac{1}{4}$, they are normalised. Hence they are idempotent, as can be verified by direct multiplication. We can also verify that their products are zero. They accordingly satisfy

$$J_a{}^2 = J_a, \quad J_aJ_b = 0, \quad J_a+J_b+J_c+J_d = 1. \tag{5·72}$$

A set of operators satisfying the conditions (5·72) is called a *spectral set*. Here the set consists of four operators only. The more familiar examples of spectral sets in physics include an infinite number of operators. For example, let J_λ denote the operation of selecting light of wave length λ from a source of light represented by ψ; thus the light of wave length λ existing in the source is represented by $J_\lambda\psi$. If we repeat the selective operation J_λ on $J_\lambda\psi$, it makes no difference; hence $J_\lambda{}^2 = J_\lambda$. The symbol $J_{\lambda'}J_\lambda$ denotes the operation of selecting wave length λ' out of light already selected as being of wave length λ; the result is obviously zero. Further, selecting every wave length in turn and adding the results, we reproduce the original source of light; hence $\Sigma_\lambda J_\lambda$ is equal to the "identical operator" 1. The selective operators of spectral analysis therefore fulfil the equations (5·72), which ensure that they are idempotent, non-overlapping and exhaustive.

As G. Temple has pointed out† it is equations of the form (5·72) which directly embody the physical conception of a "pure constituent". The mathematically convenient criterion of purity, namely factorisability of the operator in matrix representation, should be regarded as derived from (5·72) rather than *vice versa*.

This suggests a new approach to the theory of the representation of phenomena by E-symbols. We can regard the matrices E_μ as introduced by a spectral analysis of entities represented by algebraic numbers (in particular, probability distributions or densities) into four pure constituents given by (5·71). This point of view is developed in § 13·6.

5·8. The Complete Stream Vector of a Particle.

Consider a particle in spherical space-time. A classical particle is described by two 4-vectors, namely a position vector and a velocity vector. In five-dimensional representation the position vector is the radius of space-time which passes through the particle; the velocity vector is at right angles to it and lies in the four-dimensional hypersphere.

If we take axes such that the position vector is in the E_5 direction and the velocity vector is in the E_4 direction, the two vectors reduce to single components j_5 and j_4. Let us treat them as components of a single wave tensor. There is, of course, no compulsion to combine them; there is no unique definition of *the* wave tensor of a particle, any more than in ordinary relativity theory there is a unique definition of *the* tensor of a particle; and it would be legitimate to investigate a tensor representing position only or velocity only, if desired. But we shall try to find a tensor, called the *complete stream vector*, which comprises both.

If the complete stream vector is pure, it must have two more components besides j_4 and j_5. We may take it to be

$$(E_{41}+E_{50}+iE_{40}+iE_{15})\,\alpha, \tag{5·81}$$

which is of the standard form (5·56) with $\mu, \nu, \sigma, \tau = 5, 4, 1, 0$. The additional terms define an axis in the three-dimensional space, which is in some way characteristic of the particle. This axis, which in (5·81) is taken to be in the E_1 direction, is called the *spin axis*.

The question now arises whether in ascribing a complete stream vector J to the particle we should take (5·81) to be the actual vector J or the vector

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† "The Physical Principles of the Quantum Theory", *Proc. Roy. Soc.* A, **138**, 479.

density $J.iE_5$. This is answered by the Uncertainty Principle, which asserts that a particle cannot have exact position and exact velocity simultaneously. Thus our combination of a position vector and a velocity vector will not apply to a discrete particle, but describes an element of its probability distribution. We must therefore take (5·81) to be a vector density, so that

$$iJE_5 = (E_{41} + E_{50} + iE_{40} + iE_{15})\,\alpha. \qquad (5{\cdot}82)$$

From this we obtain $$J = (E_{23} + E_{16} + E_{45} + E_{01})\,\alpha, \qquad (5{\cdot}83)$$

which is of the standard form (5·54) for a non-degenerate pure tensor.

The direction of the spin axis is shown by the term E_{01}, or equivalently by the term E_{23} which gives the plane of the spin. The velocity vector, which by our choice of axes is in the time direction, is represented by the term E_{45}, this being the matrix of the rotation which would displace the particle in the time direction. Instead of a position vector, we have an invariant E_{16}. The "position" of the particle is therefore invariant for all relativity rotations; this is only possible if we represent the particle as an entity uniformly distributed throughout the hypersphere of space-time. This is in agreement with the uncertainty principle; for we have ascribed an exact velocity vector E_{45} to the particle, and therefore its position is entirely indeterminate.

The attempt to assign a combination of position vector and velocity vector to a particle breaks down, as the uncertainty principle foretells. The position vector E_{50} in (5·82) defines, not the position of the particle, but the position of an element of its probability distribution selected for consideration.

We define an *elementary particle* to be an entity whose characteristics are completely specified by a complete stream vector of the type (5·83), so that it can be represented by simple wave vectors ψ, χ^*. It has exact momentum but indeterminate position. This would perhaps more usually be called an elementary state of an elementary particle; and it is contemplated that a number of elementary states may be superposed—forming a wave packet which has approximate position and momentum. It is to be remembered, however, that the properties of observational significance are relations to other elementary particles or combinations of particles, and not the primitive relations to a symbolic frame summed up in (5·83). We must not be in too great a hurry to identify our formulae with those employed in the practical applications of quantum theory.

By (4·65) the three-dimensional vector density or strain vector corresponding to J is $S = iJE_{45}$. For the special wave tensor (5·83), we find

$$S = -J. \qquad (5{\cdot}84)$$