PART I

WAVE-TENSOR CALCULUS

CHAPTER I

TENSORS AND MATRICES

1·1. Linear Transformations.

A physical system may be described in many alternative ways. Different systems of coordinates may be used for specifying its position; different systems of units may be used for the measurement of mass, length, time; and so on. Accordingly our attention is directed to the problem of comparing systems of description in which there is a one-to-one correspondence between quantities A, B, C, ... occurring in one description and quantities A', B', C', ... occurring in another description.

The description commonly includes sets of associated quantities which are regarded as "components" of a single entity, e.g. the three components of a force. We then have a correspondence between an array of n quantities A_μ in one description and A_μ' in another description ($\mu = 1, 2, \dots n$).

We proceed at once to a special case of great importance, viz. when A_μ' is given by a linear transformation of A_μ

$$\left.\begin{aligned} A_1' &= q_{11}A_1 + q_{12}A_2 + \dots + q_{1n}A_n \\ A_2' &= q_{21}A_1 + q_{22}A_2 + \dots + q_{2n}A_n \end{aligned}\right\} \qquad (1\cdot 11)$$

etc. Using the summation convention of the tensor calculus, these formulae are written more compactly

$$A_\sigma' = q_{\sigma\mu}A_\mu \qquad (1\cdot 12)$$

and the transformation, or change of description, is described as $A_\sigma \to q_{\sigma\mu}A_\mu$.

The array of coefficients $q_{\sigma\mu}$ defines the change of the system of description, so far as the characteristic A_μ is concerned. Linear transformations possess the Group property; that is to say, the resultant of a succession of linear transformations is a linear transformation. Thus we can have a set of systems of description such that, in passing from any one description to any other, the transformation of A_μ is always linear. When for all systems of description contemplated the transformation of A_μ is linear, A_μ is called a *tensor*.

By solving equations (1·11) we can find $A_1, A_2, \dots$ in terms of $A_1', A_2', \dots$. The resulting formulae are linear and may be written

$$A_\sigma = q_{\sigma\mu}' A_\mu'. \qquad (1\cdot 13)$$

The array of coefficients $q_{\sigma\mu}'$ defines the *inverse* transformation to that defined by $q_{\sigma\mu}$.

If B_μ is another array of n quantities occurring in the description of the physical system, and in the change of description in which $A_\sigma \to q_{\sigma\mu}A_\mu$

$$B_\sigma \to q_{\sigma\mu}B_\mu, \qquad (1\cdot 14)$$

B_μ is said to be a *tensor of the same kind* as A_μ (or to be *cogredient* with A_μ).

If C_μ is another array of n quantities occurring in the description, and in the change of description in which $A_\sigma \to q_{\sigma\mu} A_\mu$

$$C_\sigma \to q_{\mu\sigma}' C_\mu, \tag{1·15}$$

C_μ is said to be a *tensor of opposite kind* to A_μ (or to be *contragredient* to A_μ). Note the inversion of the order of the suffixes of q'.

From A_μ and C_μ we may form an array of n^2 quantities $A_\mu C_\nu$ which follows the transformation law

$$A_\sigma' C_\tau' = q_{\sigma\mu} A_\mu q_{\nu\tau}' C_\nu = q_{\sigma\mu} q_{\nu\tau}' A_\mu C_\nu. \tag{1·16}$$

If $T_{\mu\nu}$ is an array of n^2 quantities occurring in the description, and in the change of description in which $A_\mu \to q_{\sigma\mu} A_\mu$

$$T_{\sigma\tau} \to q_{\sigma\mu} q_{\nu\tau}' T_{\mu\nu} \tag{1·17}$$

(i.e. if it is transformed in the same way as $A_\mu C_\nu$), then $T_{\mu\nu}$ is said to be a *mixed tensor of the second rank* of the class A_μ.

Tensor properties do not necessarily depend on the physical nature of the entity that is being described; they depend on the variety of descriptions which we admit. For example, the statement that B_μ is a tensor of the same kind as A_μ announces a limitation of the variety of description contemplated; for there can be no *compulsion* to change our description of one physical feature of the system when the description of another feature is changed. But unless there is some systematic plan underlying our descriptions it will be impossible to assert any general laws governing the quantities occurring in the descriptions.

For example, the strength of the wind is sometimes described by a number of dynes per square centimetre and sometimes by a number on the Beaufort scale. We cannot expect to find exact equations (relating our measures of the strength of the wind to other meteorological characteristics) applicable to *both* codes of measurement. By taking the wind strength to be a tensor of the class of tensors used for describing other meteorological characteristics we rule out one or other description—not as illegitimate, but as unsuited to the purpose we have in mind, viz. to express the regularities underlying natural phenomena by mathematical equations governing the quantities which occur in our descriptions of the phenomena.

1·2. Space Tensors and Wave Tensors.

When the change of system of description includes a change of coordinates from (x_1, x_2, x_3, x_4) to (x_1', x_2', x_3', x_4'), an infinitesimal coordinate difference dx_μ is transformed according to the formula

$$dx_1' = \frac{\partial x_1'}{\partial x_1} dx_1 + \frac{\partial x_1'}{\partial x_2} dx_2 + \frac{\partial x_1'}{\partial x_3} dx_3 + \frac{\partial x_1'}{\partial x_4} dx_4 \tag{1·21}$$

etc. This may be written in the form (1·15)

$$dx_\sigma' = q_{\mu\sigma}' dx_\mu \quad (q_{\mu\sigma}' = \partial x_\sigma' / \partial x_\mu). \tag{1·22}$$

Thus every change of description contemplated as admissible corresponds to a linear transformation of dx_μ. Accordingly dx_μ is a tensor; we call it a *displacement vector*.

This is the basic tensor of the class of tensors used in the ordinary tensor calculus. Displacement vectors and all tensors of the same kind are called contravariant vectors; tensors of opposite kind are called covariant vectors. Mixed tensors of the same class are defined as in (1·17); and more generally tensors of higher rank with 16, 64, 256, ... components are introduced, their transformation laws being

$$A'^{\alpha\beta\ldots}_{\gamma\delta\ldots} = q_{\mu\alpha}' q_{\nu\beta}' \ldots q_{\gamma\sigma} q_{\delta\tau} \ldots A^{\mu\nu\ldots}_{\sigma\tau\ldots}. \tag{1·23}$$

We shall call this class of tensors *space tensors*.

Thus, although the theory of tensors belongs primarily to the algebraic theory of transformations, it has usually been linked to geometry by identifying the basic tensor of the algebraic scheme with a geometrical displacement or coordinate difference dx_μ. *We shall here discard this special linkage*. We shall introduce another class of tensors called *wave tensors*, derived from a basic contravariant wave vector χ_μ in the same way that the space tensors are derived from the basic contravariant space vector dx_μ.

For the moment we leave the basic wave vector unidentified. But at a certain point in the development of the system of wave tensors, we shall be able to side-step into a new class of tensors. On examining the properties of the new tensors we shall find that they can be identified with space tensors. Thus the wave-tensor calculus leads up to the ordinary space-tensor calculus and includes it as a side branch; but its greater comprehensiveness fits it to deal with certain entities in modern quantum theory which are not describable by space tensors.

The basic wave vector will be identified in Chapter v. It turns out to be the four-valued wave symbol introduced into physics by P. A. M. Dirac in his linear wave equation of the electron. Vectors of this class cannot be reached from the ordinary calculus of space tensors, which does not begin far enough back. Our plan accordingly is to begin with these vectors, and lead up to the ordinary space vectors at a later stage.

1·3. Chain Multiplication.

Let $A_\mu{}^\nu$, $B_\mu{}^\nu$ be two mixed tensors of the second rank. Having regard to the summation convention we recognise four different products

$$A_\mu{}^\nu B_\sigma{}^\tau, \quad A_\mu{}^\nu B_\nu{}^\tau, \quad A_\mu{}^\nu B_\sigma{}^\mu, \quad A_\mu{}^\nu B_\nu{}^\mu. \tag{1·31}$$

The first is the *outer product*, and the fourth is the *inner* or *scalar product*.

The second and third are called *matrix products* and are denoted by AB and BA respectively.

Matrix products are formed by chain multiplication, i.e. the second suffix of one factor is repeated as the first suffix of the succeeding factor (the repetition introducing a summation in accordance with the summation convention). The product $A_\mu{}^\nu B_\nu{}^\tau$ is of this form. $A_\mu{}^\nu B_\sigma{}^\mu$ is not a chain product as it stands; but it becomes one if it is rewritten as $B_\sigma{}^\mu A_\mu{}^\nu$.

On the understanding that chain multiplication is the only kind of multiplication admitted, no suffixes need appear in the formulae, since the reader can always supply appropriate suffixes when required. Thus the product of a number of double-suffixed quantities is written

$$P = ABCD, \tag{1·321}$$

which stands for

$$P_\mu{}^\nu = A_\mu{}^\alpha B_\alpha{}^\beta C_\beta{}^\gamma D_\gamma{}^\nu. \tag{1·322}$$

This rule of multiplication is the distinctive feature of the matrix calculus. The notation is so useful that we cannot afford to do without it. Nevertheless matrix calculus suffers from being more limited than tensor calculus; and we often want to introduce outer and scalar products and other combinations for which matrix calculus provides no notation. This necessitates resorting to various awkward shifts, and occasionally reverting to the full suffixed expressions.

Chain multiplication does not contemplate quantities with more than two suffixes. We shall at first limit the term "matrix" to two-suffixed quantities representing two-dimensional arrays. Technically one-dimensional arrays are also matrices, but it would probably be confusing to include them. One-dimensional arrays will here be called vectors, even when no question of transformation properties arises. The term implies very little restriction so long as we do not specify the *kind* of vector.

Chain multiplication cannot be carried beyond a vector, so that vectors can only occur at the beginning or end of a matrix product. We shall distinguish initial vectors by an asterisk, final vectors being unmarked. This notation allows us to reintroduce outer multiplication to a limited extent. The rule is that, if it is impossible to interpret two symbols in juxtaposition as a chain product, they are to be interpreted as an outer product. Thus if ψ_μ is a vector, the expressions

$$A\psi B, \quad A\psi^* B$$

are interpreted as

$$(A\psi) \times B, \quad A \times (\psi^* B),$$

where the symbol $\times$ indicates outer multiplication, chain multiplication being impossible after a final vector or before an initial vector. Or, with suffixes,

$$A\psi B = A_\mu{}^\nu \psi_\nu B_\sigma{}^\tau, \quad A\psi^* B = A_\mu{}^\nu \psi^\sigma B_\sigma{}^\tau.$$

In particular, we have the following notation which is of great importance

$$\left.\begin{aligned}\psi\chi^* &\text{ denotes the outer product } \psi_\mu \chi^\nu,\\ \chi^*\psi &\text{ denotes the scalar product } \chi^\mu \psi_\mu.\end{aligned}\right\} \qquad (1·33)$$

The asterisk is a substitute for suffix indications, and is dropped when the suffixes are inserted.

A feature of matrix multiplication is that it is non-commutative; that is to say

$$BA \neq AB. \qquad (1·34)$$

It is to be remembered that the non-commutation only arises through the omission of suffixes; when suffixes are inserted in BA, the factors commute as usual. Thus

$$B_\mu{}^\alpha A_\alpha{}^\nu = A_\alpha{}^\nu B_\mu{}^\alpha. \qquad (1·35)$$

Since the suffixes are often omitted, we can no longer depend on discriminating contravariant from covariant vectors by the upper and lower positions of the suffixes. There would be little advantage in retaining a method of discrimination which only worked spasmodically. Accordingly, we shall in future generally write all wave tensor suffixes in the lower position.

1·4. Transformation Laws of Wave Tensors.

In § 1·1 we introduced three kinds of tensors of the class A_μ with transformation laws (1·14), (1·15) and (1·17) respectively. The formulae may be written as

$$B_\sigma' = q_{\sigma\mu} B_\mu, \quad C_\sigma' = C_\mu q_{\mu\sigma}', \quad T_{\sigma\tau}' = q_{\sigma\mu} T_{\mu\nu} q_{\nu\tau}'.$$

The products, as here written, are all chain products, so that the suffixes may be omitted and we have

$$B' = qB, \quad C^{*\prime} = C^* q', \quad T' = qTq'. \qquad (1·41)$$

Further by (1·12) and (1·13)

$$A_\sigma' = q_{\sigma\mu} A_\mu, \quad A_\mu = q_{\mu\tau}' A_\tau'.$$

Therefore

$$A_\sigma' = q_{\sigma\mu} q_{\mu\tau}' A_\tau'. \qquad (1·421)$$

But

$$A_\sigma' = \delta_{\sigma\tau} A_\tau', \qquad (1·422)$$

where $\delta_{\sigma\tau}$ is the substitution operator, viz.

$$\left.\begin{aligned}\delta_{\sigma\tau} &= 1, \quad \text{if } \sigma = \tau\\ &= 0, \quad \text{if } \sigma \neq \tau\end{aligned}\right\}. \qquad (1·43)$$

Since A_τ' is an arbitrary array of four numbers, it follows from (1·421) and (1·422) that

$$q_{\sigma\mu} q_{\mu\tau}' = \delta_{\sigma\tau}. \qquad (1·44)$$

The left-hand side is a chain product; we can therefore drop the suffixes, obtaining

$$qq' = \delta. \qquad (1·45)$$

In matrix calculus δ has the algebraic properties of the number 1. For, if S is any matrix
$$\delta_{\mu\alpha}S_{\alpha\nu}=S_{\mu\nu}, \quad S_{\mu\alpha}\delta_{\alpha\nu}=S_{\mu\nu},$$
so that, dropping suffixes, $\delta S=S, \quad S\delta=S.$

Accordingly δ is called the *unit matrix*; and since it is equivalent to the number 1 in matrix calculus, we shall often denote it by 1, or with suffixes $(1)_{\mu\nu}$. Then (1·45) becomes $qq'=1$. Thus q' may be called the reciprocal of q, and it will sometimes be written as q^{-1}.

The formulae (1·41) and (1·45) constitute the principal transformation formulae in wave-tensor calculus. Summarising our results for reference, and changing to the notation which we shall usually employ, we have the following classification and nomenclature:

Covariant (final) wave vectors
$$\psi'=q\psi. \tag{1·461}$$

Contravariant (initial) wave vectors
$$\chi^{*\prime}=\chi^{*}q'. \tag{1·462}$$

Mixed wave tensors
$$T'=qTq', \tag{1·463}$$
with
$$qq'=1. \tag{1·464}$$

These would reduce to the transformation laws of the ordinary tensor calculus if we set
$$q_{\sigma\mu}=\partial x_{\mu}/\partial x_{\sigma}', \quad q_{\sigma\mu}'=\partial x_{\mu}'/\partial x_{\sigma}. \tag{1·47}$$

But, as already explained, the wave tensors are not linked to geometry in this way, and (1·47) does not apply. For a transformation of wave tensors, any matrix which has a reciprocal may be used as q; that is to say, the corresponding transformation will give a new description which is included in the whole group of descriptions contemplated.

A matrix which has no reciprocal is said to be *singular*. A singular matrix may be regarded as a generalisation of the algebraic number 0 in much the same way that the unit matrix is a generalisation of the number 1; but there are infinitely many different singular matrices. As q approaches a singular value, one or more elements of its reciprocal q' tend to infinity; singular matrices q are therefore excluded in the foregoing transformation theory.

1·5. Initial and Final Wave Vectors.

The terms "initial" and "final", applied to wave vectors, define their behaviour in regard to chain multiplication, and do not necessarily describe their actual position in the sequence of factors (cf. (1·33)). As far as Chapter VI (inclusive) the initial vectors will be contravariant and the final vectors covariant. But it must not be supposed that this is a general rule, or that the asterisk is a symbol for contravariance.

In order to express the covariant transformation law (1·461) in a form appropriate for an initial covariant vector, we introduce a matrix $\bar{q}$ which is the *transpose* of q, obtained by interchanging rows and columns; thus

$$\bar{q}_{\alpha\beta} = q_{\beta\alpha}. \tag{1·51}$$

Then (1·461) stands for

$$\psi_\alpha{}' = q_{\alpha\beta}\psi_\beta = \bar{q}_{\beta\alpha}\psi_\beta = \psi_\beta \bar{q}_{\beta\alpha}.$$

Hence, dropping suffixes, $\psi^{*\prime} = \psi^* \bar{q}$. Treating (1·462) similarly, we have the transformation laws:

Initial covariant wave vectors

$$\psi^{*\prime} = \psi^* \bar{q}, \tag{1·521}$$

Final contravariant wave vectors

$$\chi' = \bar{q}'\chi. \tag{1·522}$$

The outer product $\psi\phi^*$ of two covariant wave vectors ψ, ϕ is a covariant wave tensor S. Using (1·461) and (1·521) we obtain the transformation law:

Covariant wave tensors $\qquad S' = qS\bar{q}. \qquad$ (1·53)

These formulae will not be required until Chapter VII.